# Factorization of Kazhdan–Lusztig elements for Grassmanians

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Abstract. We show that the Kazhdan-Lusztig basis elements $C_w$ of the Hecke algebra of the symmetric group, when $w \in S_n$ corresponds to a Schubert subvariety of a Grassmann variety, can be written as a product of factors of the form $T_i + f_j(v)$, where $f_j$ are rational functions.

## 1. Notation

In this section, we briefly list the main facts and notations related to Kazhdan–Lusztig polynomials and their parabolic analogues (see [**D**], [**S**]). We use the following notations:

$\mathcal{H}$—the Hecke algebra of the symmetric group $S_n$; we consider it as an algebra over the field $\mathbf{Q}(v)$ (the variable $v$ is related to the variable $q$ used by Kazhdan and Lusztig via $v = q^{1/2}$), and we write the quadratic relation in the form

$$(T_i - v)(T_i + v^{-1}) = 0.$$

$C_w$—KL basis in $\mathcal{H}$, which we define by the conditions $\overline{C_w} = C_w$, $C_w - T_w \in \oplus v\mathbf{Z}[v]T_y$.

For any subset $J \subset \{1, \dots, n-1\}$, we denote by $W_J \subset S_n$ the corresponding parabolic subgroup, and by $W^J$ the set of minimal length representatives of cosets $S_n/W_J$. We also denote by $M^J$ the $\mathcal{H}$-module induced from the one-dimensional representation of $\mathcal{H}(W_J)$, given by $T_j m_1 = -v^{-1}m_1, j \in J$. We denote $m_y = T_y m_1, y \in W^J$ the usual basis in $M^J$.

We define the parabolic KL basis $C_y^J, y \in W^J$ in $M^J$ by $\overline{C_y^J} = C_y^J, C_y^J - m_y \in \oplus_{z \in W^J} v\mathbf{Z}[v]m_z$.

Denote for brevity $C_J = C_{w_0^J}$ the element of KL basis in $\mathcal{H}$ corresponding to the element of $w_0^J$ of maximal length in $W_J$. The following result is well-known (see, e.g., [**S**]).

Lemma 1. (i)

$$C_J = \sum_{w \in W^J} (-v)^{l(w_0^J) - l(w)} T_w.$$

(ii) *Let $w \in W$ be such that it is an element of maximal length in the coset $wW_J$ (which is equivalent to $w = \tau w_0^J$ for some $\tau \in W^J$). Then $C_w = XC_J$ for some $X \in \oplus_{y \in W^J} \mathbf{Z}[v^{\pm 1}]T_y$.*

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(iii) *Let* $X \in \oplus_{y \in W^J} \mathbf{Z}[v^{\pm 1}] T_y$. *Then*

$$Xm_1 = C^J_\tau \iff XC_J = C_{\tau w_0^J}.$$

Let us now consider the special case of the above situation. From now on, fix $k \le n-1$, and let $J = \{1, \dots, k-1, k+1, \dots, n-1\}$ so that $W_J = S_k \times S_{n-k}$ is a maximal parabolic subgroup in $S_n$. In this case, the module $M^J$ can be described as follows:

$$
\begin{aligned}
&M = \bigoplus_{\varepsilon \in E} \mathbf{Q}(v)\varepsilon,\\
&T_i \varepsilon = \begin{cases} s_i\varepsilon, & (\varepsilon_i, \varepsilon_{i+1}) = (+-),\\ -v^{-1}\varepsilon, & (\varepsilon_i, \varepsilon_{i+1}) = (--) \text{ or } (++),\\ s_i\varepsilon + (v - v^{-1})\varepsilon, & (\varepsilon_i, \varepsilon_{i+1}) = (-+), \end{cases}
\end{aligned}
\tag{1}
$$

where $E$ is the set of all length $n$ sequences of pluses and minuses which contain exactly $k$ pluses. The relation of this with the previous notation is given by $m_y \leftrightarrow y(\mathbf{1}) = T_y(\mathbf{1})$, where

$$\mathbf{1} = (\underbrace{+ \cdots +}_{k} \underbrace{- \cdots -}_{n-k}). \tag{2}$$

In particular, $m_1 \leftrightarrow \mathbf{1}$.

The set of minimal length representatives $W^J$ also admits a description in terms of Young diagrams. Namely, let $\lambda$ be a Young diagram which fits inside the $k \times (n-k)$ rectangle. Define $w_\lambda \in S_n$ by

$$w_\lambda = \prod_{(i,j) \in \lambda} s_{k+j-i}, \tag{3}$$

where $(i, j)$ stands for the box in the $i$-th row and $j$-th column, and the product is taken in the following order: we start with the lower right corner and continue along the row, until we get to the first column; then we repeat the same with the next row, and so on until we reach the upper left corner.

Example 1. Let $\lambda$ be the diagram shown below, and $k = 7$ (to assist the reader, we put the numbers $k + j - i$ in the diagram).

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
7 & 8 & 9 & 10 & 11 & 12 \\
\hline
6 & 7 & 8 \\
\cline{1-3}
5 & 6 & 7 \\
\cline{1-3}
4 \\
\cline{1-1}
3 \\
\cline{1-1}
\end{array}
$$

Then $w_\lambda = s_3 \cdot s_4 \cdot s_7 s_6 s_5 \cdot s_8 s_7 s_6 \cdot s_{12} s_{11} s_{10} s_9 s_8 s_7$ (for easier reading, we separated products corresponding to different rows by $\cdot$).

The proof of the following proposition is straightforward.

Proposition 2. *The corespondence* $\lambda \mapsto w_\lambda$, *where* $w_\lambda$ *is defined by* (3), *is a bijection between the set of all Young diagrams which fit inside the* $k \times (n-k)$ *rectangle and* $W^J$.

## 2. The main theorem

As before, we fix $k \le n-1$ and let $J = \{1, \dots, k-1, k+1, \dots, n-1\}$. Unless otherwise specified, we only use Young diagrams which fit inside the $k \times (n-k)$ rectangle.

For a Young diagram $\lambda$, we define the shifts $r_{i,j} \in \mathbf{Z}_{>0}, (i,j) \in \lambda$ by the following relation

$$r_{ij} = \max(r_{i,j+1}, r_{i+1,j}) + 1, \tag{4}$$

where we let $r_{ij} = 0$ if $(i,j) \notin \lambda$.

Example 2. For the diagram $\lambda$ from Example 1, the shifts $r_{ij}$ are shown below.

| | | | | | |
|---|---|---|---|---|---|
| 6 | 5 | 4 | 3 | 2 | 1 |
| 4 | 3 | 2 | | | |
| 3 | 2 | 1 | | | |
| 2 | | | | | |
| 1 | | | | | |

Next, let us define for each diagram $\lambda$ an element $X_\lambda \in \mathcal{H}$ by

$$X_\lambda = \prod_{(i,j)\in\lambda} \left( T_{k+j-i} - \frac{v^{r_{ij}}}{[r_{ij}]} \right) \tag{5}$$

where, as usual, $[r] = (v^r - v^{-r})/(v - v^{-1})$, and the product is taken in the same order as in (3).

The main result of this paper is the following theorem.

Theorem 3. *Let $\lambda$ be a Young diagram. Then*

$$X_\lambda \mathbf{1} = C^J_{w_\lambda}.$$

Note that by Lemma 1, this is equivalent to

$$X_\lambda C_J = C_{w_\lambda w_0^J}. \tag{6}$$

We remind the reader that the Kazhdan-Lusztig elements $C_{ww_0^J}$, where $w \in W^J$, and $W_J$ is a maximal parabolic in $S_n$ (they are also known as KL elements for Grassmanians), have been studied in a number of papers. A combinatorial description was given in [**LS1**]; it was interpreted geometrically in [**Z**], and in terms of representations of quantum $\mathfrak{gl}_m$ in [**FKK**]. However, it is unclear how these results are related with the factorization given by the theorem above. A similar factorization was given in [**L**] for those permutations which correspond to non-singular Schubert varieties—i.e., for those $w$ such that, for any $v \in S_n$, the Kazhdan-Lusztig polynomial $P_{v,w}$ is either 1 or 0.

Note that one can easily check that the elements $X_\lambda$ are invariant under the Kazhdan–Lusztig involution: $\overline{X_\lambda} = X_\lambda$; thus, all the difficulty is in proving that they are integral and have the right specialization at $v = 0$.

A crucial step in proving this theorem is the following proposition.

Proposition 4. *Theorem 3 holds when $\lambda$ is the $k \times (n-k)$ rectangle.*

Proof. For any $w \in S_n$, choose a reduced expression $w = s_{i_\ell} \dots s_{i_1}$. Define the element $\nabla_w \in \mathcal{H}$ by

$$\nabla_w = \left( T_{i_\ell} - \frac{v^{r_\ell}}{[r_\ell]} \right) \dots (T_{i_1} - v), \tag{7}$$

where $r_1, \dots, r_\ell \in \mathbf{Z}_+$ are defined as follows: if $s_{i_{m-1}} \dots s_{i_1}(1, \dots, n) = (\dots, a, b, \dots)$ (in $i_m$-th, $(i_m+1)$-st places), then $r_m = b-a$. Then $\{\nabla_w, w \in S_n\}$ is a Yang-Baxter basis of the Hecke algebra, and we have (see [**DKLLST**, §3]):

LEMMA 5. (i) *The element $\nabla_w$ does not depend on the choice of reduced expression.*

(ii) *If $w_0^J$ is the longest element in some parabolic subgroup $W_J \subset S_n$, then $\nabla_{w_0^J} = C_J$.*

Now, let us prove our proposition, i.e. that $X_\lambda C_J$ is a KL element for rectangular $\lambda$. In this case, $w_\lambda$ is the longest element in $W^J$:

$$w_\lambda(\mathbf{1}) = (\underbrace{- \cdots -}_{n-k}\underbrace{+ \cdots +}_{k}).$$

Let us choose the following reduced expression for the longest element $w_0$ in $S_n$: $w_0 = w_\lambda w_0^J$, where we take for $w_\lambda$ the reduced expression given by (3). Then one easily sees that definition (7) in this case gives

$$\nabla_{w_0} = X_\lambda \nabla_{w_0^J}.$$

By Lemma 5, we get $C_{w_0} = X_\lambda C_J$, which is exactly the statement of the proposition. $\square$

The proof in the general case is based on the following proposition. Denote

$$O(v^m) = \{f \in \mathbf{Q}(v) | f \text{ has zero of order} \geq m \text{ at } v = 0\}. \tag{8}$$

PROPOSITION 6.

$$X_\lambda \mathbf{1} = w_\lambda(\mathbf{1}) + \sum_{\varepsilon \in E} O(v)\varepsilon.$$

A proof of this proposition is given in Section 3.

Now we can give a proof of the main theorem. First, one easily checks the invariance under the bar involution, since

$$\overline{T_i - \frac{v^r}{[r]}} = T_i - \frac{v^r}{[r]}.$$

Combining this with Proposition 6, we see that it remains to show that $X_\lambda C_J$ are integral, i.e. $X_\lambda C_J \in \oplus \mathbf{Z}[v^{\pm 1}]T_w$ (note that it is not true that $X_\lambda$ itself is integral.) This will be done by induction.

Let $\lambda$ be a Young diagram. Then we claim that any such diagram can be presented as a union $\lambda = \lambda' \sqcup \mu$, where $\mu$ is a rectangle, and $\lambda'$ is again a Young diagram such that for $(i,j) \in \lambda'$, the shifts $r^{\lambda'}_{(i,j)} = r^{\lambda}_{(i,j)}$. It can be formally proved as follows: if one writes the successive widths and heights of the stairs of the diagram

$$\infty, (a_1, b_1), (a_2, b_2), ...(a_k, b_k), \infty$$

then there is at least one index $i$ for which $a_i \leq b_{i+1}$ and $b_i \leq a_{i+1}$. In that case, the rectangle $\mu$ has the lower right corner $i$.

EXAMPLE 3. For the diagram $\lambda$ from Example 1, the sequence $(a_k, b_k)$ is given by $\infty, (1,2), (2,2), (3,1), \infty$, and the subdiagram $\mu$ is the shaded $2 \times 2$ square, as

shown below. As before, we also included the shifts $r_{ij}$ in this diagram. The subsets $I^\mu, J^\mu$ in this case are given by $I^\mu = \{6, 7, 8\}$, $J^\mu = \{6, 8\}$.

| | | | | | |
|---|---|---|---|---|---|
| 6 | 5 | 4 | 3 | 2 | 1 |
| 4 | 3 | 2 | | | |
| 3 | 2 | 1 | | | |
| 2 | | | | | |
| 1 | | | | | |

Let us choose for $\lambda$ the presentation $\lambda = \lambda' \sqcup \mu$, where $\mu$ is a rectangle, as above. Then $X_\lambda = X_\mu X_{\lambda'}$.

Define the subsets $I^\mu, J^\mu \subset \{1, \dots, n-1\}$ by $I^\mu = \{k'-a+1, \dots, k'+b-1\}$, $J^\mu = I^\mu \setminus \{k'\}$, where $k' = k - i + j$, $(i, j)$—coordinates of the UL corner of $\mu$, $a$ and $b$ are numbers of rows and columns in $\mu$ respectively.

We need to show that $X_\mu X_{\lambda'} C_J \in \sum \mathbf{Z}[v^{\pm 1}] T_y$. By induction assumption, we may assume that $X_{\lambda'} C_J = C_\sigma$, where we denoted for brevity $\sigma = w_{\lambda'} w_0^J$. It is easy to show that if $\mu$ is chosen as before, then $\sigma$ is the maximal length element in the coset $W_{J^\mu}\sigma$. Thus, by Lemma 1, we can write $C_\sigma = C_{J^\mu} Y$ for some integral $Y \in \mathcal{H}$. Therefore, $X_\mu X_{\lambda'} C_J = X_\mu C_{J^\mu} Y$. Since $W_{I^\mu}$ is itself a symmetric group, and $W_{J_\mu}$ is a maximal parabolic subgroup in it, we can use Proposition 4, which gives $X_\mu C_{J^\mu} = C_{I^\mu}$, and therefore, $X_\mu X_{\lambda'} C_J = C_{I^\mu} Y \in \sum \mathbf{Z}[v^{\pm 1}] T_w$. □

## 3. Proof of regularity at $v = 0$

In this section we give the proof of Proposition 6. Before doing so, let us introduce some notation.

As before, assume that we are given $n, k, \lambda$ and a collection of positive integers $r_{ij}, (i, j) \in \lambda$ (not necessarily defined as in (4)). Let $\varepsilon \in E$ be a sequence of pluses and minuses. We define the *weight* $r_\lambda(\varepsilon)$ as folllows.

Define $a(i), i = 1 \dots k$ by $a(i) = k + \lambda_i - i + 1$. Equivalently, these numbers can be characterized by saying that $w_\lambda(\mathbf{1})$ has pluses exactly at positions $a(k), \dots, a(1)$.

Define $r_\lambda(\varepsilon) = \sum_{t=1}^{n} r_t(\varepsilon)$, where $r_t(\varepsilon)$ is defined as follows:

(i) if $t = a(i), \varepsilon_t = -$ then $r_t(\varepsilon) = r_{i,\lambda_i} - 1$

(ii) if $a(i) < t < a(i + 1), \varepsilon_t = +$ then $r_t(\varepsilon) = r_{i,j}, k + j - i = t$

(iii) otherwise, $r_t(\varepsilon) = 0$

In a sense, $r_\lambda(\varepsilon)$ measures the discrepancy between $\varepsilon$ and $w_\lambda(\mathbf{1})$. Indeed, let us denote the numbers of rows and columns in $\lambda$ by $i, j$ respectively, and let $\varepsilon$ be such that

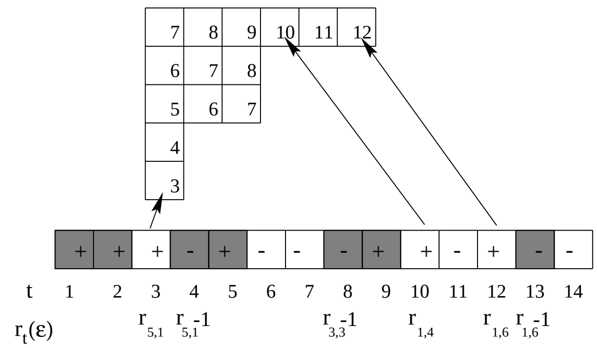
$$\begin{aligned} \varepsilon_t &= + \text{ for } t \le k - i, \\ \varepsilon_t &= - \text{ for } t > k + j. \end{aligned} \tag{9}$$

Then one easily sees that

$$r_\lambda(\varepsilon) \ge 0, \qquad r_\lambda(\varepsilon) = 0 \iff \varepsilon = w_\lambda(\mathbf{1}) \tag{10}$$

Example 4. Below we illustrate the calculation of $r_\lambda(\varepsilon)$, where $\lambda$ is the diagram used in Example 1. The positions $a(i)$ are shaded (thus, the sequence of colors encodes $w_\lambda(\mathbf{1})$, with "shaded"$\leftrightarrow +$, "unshaded"$\leftrightarrow -$), and we connected unshaded pluses with the corresponding box $(i, j)$, defined in (ii) above. For convenience of

the reader, we also put the numbers $k+j-i$ (not the shifts $r_{ij}$!) in the diagram.




LEMMA 7. *Let $\lambda$ be any Young diagram inside the $k\times(n-k)$ rectangle, and let $r_{ij},(i,j)\in\lambda$, be positive integers satisfying $r_{ij}>r_{i,j+1}, r_{ij}>r_{i+1,j}$. Define $\mathcal{L}_\lambda\subset M^J$ by*

$$\mathcal{L}_\lambda=\sum_{\varepsilon\in E}O(v^{r_\lambda(\varepsilon)})\varepsilon.$$

*Then*

$$X_\lambda\mathbf{1}\in\mathcal{L}_\lambda.$$

Before proving this lemma note that due to (10), this lemma immediately implies Proposition 6.

PROOF. The proof is by induction. Let $(i,j)$ be a corner of $\lambda$, and $\lambda'=\lambda-(i,j)$, so that $X_\lambda=\left(T_{k-i+j}-\frac{v^{r_{ij}}}{[r_{ij}]}\right)X_{\lambda'}$. Since $\frac{v^r}{[r]}\in O(v^{2r-1})$, it suffices to prove that $\left(T_{k-i+j}+O(v^{2r_{ij}-1})\right)\mathcal{L}_{\lambda'}\subset\mathcal{L}_\lambda$. Since this operation only changes $\varepsilon_a,\varepsilon_{a+1}$ $(a=k-i+j)$, we need to consider 4 cases: $(++),(+-),(-+),(--)$. This is done explicitly. For example, for the $(+-)$ case, we have

$$\big(T_m+O(v^{2r_{ij}-1})\big)(\dots+-\dots)=(\dots-+\dots)+O(v^{2r_{ij}-1})(\dots+-\dots)$$

In this case, the first summand has the same weight and comes with the same power of $v$ as the original $\varepsilon$ (note that in the original $\varepsilon$, this $(+-)$ didn't contribute to the weight), so it is in $\mathcal{L}_\lambda$. As for the second summand, its weight is increased by $2r_{ij}-1$ (the plus contributes $r$ and the minus, $r-1$), but it comes with the factor $O(v^{2r_{ij}-1})$, so again, it is in $\mathcal{L}_\lambda$. The other cases are treated similarly. □

## 4. Divided differences and parabolic Kazhdan-Lusztig bases

In this section, we give a factorization for the dual Kazhdan–Lusztig basis for Grassmanians.

To induce a parabolic module, one can start from the 1-dimensional representation $T_j\mapsto v$ instead of $T_j\mapsto -1/v$ which was used in §1. We now denote the corresponding module by $M'$ and its Kazhdan-Lusztig basis by $C'^J_y$ to distinguish from previous case. Note that there exists a natural pairing between $M$ and $M'$, and $C^J_y$ and $C'^J_y$ are dual bases with respect to this pairing (see, e.g., [**S**], [**FKK**]). However, we will not use this pairing.

A simple element $T_i - v$ acts now by

$$
M' = \bigoplus_{\varepsilon \in E} \mathbf{Q}(v)\varepsilon,
$$

$$
(T_i - v)\varepsilon = \begin{cases} s_i\varepsilon - v\varepsilon, & (\varepsilon_i, \varepsilon_{i+1}) = (+-), \\ 0, & (\varepsilon_i, \varepsilon_{i+1}) = (--) \text{ or } (++), \\ s_i\varepsilon - v^{-1}\varepsilon, & (\varepsilon_i, \varepsilon_{i+1}) = (-+). \end{cases} \tag{11}
$$

Consider the space $\mathcal{P}(k, n)$ of polynomials in $x_1, \ldots, x_n$ of total degree $n-k$, and of degree at most 1 in each $x_i$. For any partition $\lambda$, denote by $x^{[\lambda]}$ the monomial $w_\lambda(x_{k+1}\cdots x_n)$, the symmetric group acting now by permutation of the $x_i$. In other words, if $w_\lambda(\mathbf{1}) = (\varepsilon_1, \ldots, \varepsilon_n)$, then $x^{[\lambda]}$ is the product of the $x_i$'s for those $i$ such that $\varepsilon_i = -$.

Consider the isomorphism of vector spaces

$$
\begin{gathered} M' \simeq \mathcal{P}(k, n) \\ w_\lambda(\mathbf{1}) \mapsto v^{-|\lambda|}x^{[\lambda]}. \end{gathered} \tag{12}
$$

Then $T_i - v$ induces the operator $\nabla_i$, acting only on $x_i, x_{i+1}$ as follows:

$$
\begin{cases} \nabla_i(x_i) = vx_{i+1} - v^{-1}x_i, \\ \nabla_i(1) = \nabla_i(x_ix_{i+1}) = 0, \\ \nabla_i(x_{i+1}) = -vx_{i+1} + v^{-1}x_i, \end{cases} \tag{13}
$$

Therefore $\nabla_i$ is the operator

$$
f \mapsto (vx_{i+1} - v^{-1}x_i)\,\partial_i(f)
$$

denoting by $\partial_i$ the divided difference

$$
f \mapsto \frac{f - f^{s_i}}{x_i - x_{i+1}}
$$

(for a more general action of the Hecke algebra on the ring of polynomials, see [**LS2**], [**DKLLST**]).

We intend to show that divided differences easily furnish the Kazhdan-Lusztig basis of $\mathcal{P}(k, n)$ (i.e. the image of the Kazhdan-Lusztig basis $C'_y, y \in W^J$ of $M'$).

To any element $\varepsilon := w_\lambda(\mathbf{1})$ of $E$ one associates a polynomial $Q_\varepsilon$ as follows

1) pair recursively $-, +$ (as one pairs opening and closing parentheses)

2) replace each pair $(-, +)$, where $-$ is in position $i$ and $+$ in position $j$, with a $x_i - v^{j+1-i}x_j$

3) replace each single $-$, in position $i$, by $x_i$

The product of all these factors by $v^{-|\lambda|}$, where $|\lambda| = \lambda_1 + \lambda_2 + \cdots$, is by definition $Q_\varepsilon$.

Theorem 8. *Let $E$ be the set of sequences of $(+, -)$ of length $n$ with $k$ pluses. Then the collection of polynomials $Q_\varepsilon$, $\varepsilon \in E$, is the Kazhdan-Lusztig basis of the space $\mathcal{P}(k, n)$.*

Proof. We shall show that

$$
Q_\varepsilon = \nabla_j \cdots \nabla_h(x_1 \cdots x_k)
$$

when $\varepsilon = w_\lambda(\mathbf{1})$, and when $s_j \cdots s_h$ is a reduced decomposition of $w_\lambda$. Now, it is clear that the inverse image of $Q_\varepsilon$ in $M'$ is invariant under involution, and it is easy to check the powers of $v$ to get that for $v = 0$, it specializes to $\varepsilon$.

Assume by induction that we already know $Q_\varepsilon$. Let us add on the right of $\varepsilon$ sufficiently many pluses, so that all minuses are now paired (the original polynomial is recovered from the new one by specializing $x_{n+1}, x_{n+2}, \ldots$ to 0). Take now any simple transposition $s_i$ such that $\varepsilon_i = +, \varepsilon_{i+1} = -$. The variables $x_i, x_{i+1}$ involve two or one factor in $P_\varepsilon$, depending whether $\varepsilon_i$ is paired or not. The only possible cases for those factors and their images under $\nabla_i$ are

$$(x_{i-a} - v^{a+1}x_i)(x_{i+1} - v^{b+1}x_{i+b+1}) \mapsto (x_{i-a} - v^{a+b+2}x_{i+b+1})(v^{-1}x_i - vx_{i+1})$$
$$(x_{i+1} - v^{b+1}x_{i+b+1}) \mapsto (v^{-1}x_i - vx_{i+1})$$

but now the new pairing of $-, +$ differs from the previous one exactly in the places described by the factors on the right. $\square$

Corollary 9. *Let $\sigma_j \cdots \sigma_h$ be a reduced decomposition of $w \in W^J$. Then the corresponding Kazhdan-Lusztig element $C'^J_w \in M'$ is equal to $(T_j - v) \cdots (T_h - v)(\mathbf{1})$.*

This factorization is equivalent to the one given in [**FKK**, Theorem 3.1]. One can check on examples that this factorization is compatible, via the duality between the two modules $M$ and $M'$, with the factorization given by Theorem 3. However, deducing Theorem 3 from Theorem 8 seems more intricate than proving the two factorization properties directly.

Example 5. Let $\lambda = [5, 3, 2]$ and $\mu = [5, 3, 3]$. Then one has

| *places* | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $w_\lambda(\mathbf{1})$ | $+$ | $-$ | $-$ | $+$ | $-$ | $+$ | $-$ | $-$ | $+$ |
| | $+$ | $-$ | | | | | $-$ | | |
| *pairing* | | | $-$ | $+$ | $-$ | $+$ | | $-$ | $+$ |
| *polynomial* | | $x_2$ | | | | | $x_7$ | | |
| | | | $(x_3$ | $-v^2x_4)$ | $(x_5$ | $-v^2x_6)$ | | $(x_8$ | $-v^2x_9)$ |
| $w_\mu(\mathbf{1})$ | $+$ | $-$ | $-$ | $-$ | $+$ | $+$ | $-$ | $-$ | $+$ |
| | $+$ | $-$ | | | | | $-$ | | |
| *pairing* | | | $-$ | | | $+$ | | | |
| | | | | $-$ | $+$ | | | $-$ | $+$ |
| *polynomial* | | $x_2$ | | | | | $x_7$ | | |
| | | | $x_3$ | | | $-v^4x_6$ | | | |
| | | | | $(x_4$ | $-v^2x_5)$ | | | $(x_8$ | $-v^2x_9)$ |

and thus

$$\begin{aligned} Q_{w_\lambda(\mathbf{1})} &= v^{-10}\, x_2x_7\,(x_3 - v^2x_4)\,(x_5 - v^2x_6)\,(x_8 - v^2x_9) \\ Q_{w_\mu(\mathbf{1})} &= v^{-11}\, x_2x_7\,(x_3 - v^4x_6)\,(x_4 - v^2x_5)\,(x_8 - v^2x_9). \end{aligned} \tag{14}$$

Note that the pairing between $-, +$, which was a key point in the description of Kazhdan-Lusztig polynomials for Grassmannians in [**LS1**], is provided by divided differences, starting from the monomial $x_{k+1} \cdots x_n$.

## References


[D] V. V. Deodhar, *On some geometric aspects of Bruhat orderings* II. *The parabolic analogue of Kazhdan–Lusztig polynomials*, J. of Algebra **111** (1987), 483–506.

[DKLLST] G. Duchamp, D. Krob, A. Lascoux, B. Leclerc, T. Scharf, and J.-Y. Thibon, *Euler–Poincaré characteristic and polynomial representations of Iwahori–Hecke algebras*, Publ. RIMS, **31** (1995), 179–201.

[FKK] I. B. Frenkel, M. G. Khovanov, and A. A. Kirillov, Jr., *Kazhdan–Lusztig polynomials and canonical basis*, Transf. Groups **3** (1998), 321–336.

[L] A. Lascoux, *Ordonner le groupe symétrique: pourquoi utiliser l'algèbre de Iwahori-Hecke?*, ICM Berlin 1998, Documenta Mathematica, vol. III (1998), 355–364.

[LS1] A. Lascoux, M.–P. Schützenberger, *Polynômes de Kazhdan & Lusztig pour les grassmaniennes*, Astérisque **87–88** (1981), 249–266.

[LS2] ———, *Symmetrization operators on polynomial rings*, Functional Anal. Appl., **21** (1987), 77–78.

[S] W. Soergel, *Kazhdan–Lusztig polynomials and combinatorics for tilting modules*, Represent. Theory (electronic journal), **1** (1997), 83–114.

[Z] A. V. Zelevinski, *Small resolutions of singularities of Schubert varieties*, Functional Anal. Appl., **17** (1983), 142–144.



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